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BALANCING FAMILIES OF INTEGER SEQUENCES

by

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In this paper we prove the following theorem: Given a sequence $A_1, A_2, \ldots; A_k = \{a_1^{(k)} < a_2^{(k)} < \ldots\}$ of infinite sets of positive integers, there exists a suitable function $g(n) = \pm 1$ for which

$$\max_m \left|\sum_{i=1}^m g(a_i^{(k)})\right| < k^{(1+\varepsilon)\log k/2} \quad \text{if} \quad k \geqq k_0(\varepsilon).$$

Some generalizations are also considered.

1. Introduction

Throughout this paper log will denote binary logarithms. Let $|\mathbf{v}|_\infty$ denote the maximum norm of the vector $\mathbf{v}$, that is, $|\mathbf{v}|_\infty = \max_i |v^{(i)}|$ where $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots)$. $\mathbf{N}$ denotes the set of positive integers.

Cantor, Erdős, Schreiber and Straus [2] (see also [3]) observed that there is a function $g\colon \mathbf{N} \to \{-1, 1\}$ for which

$$\max_{a,m} \left|\sum_{k=1}^m g(a+kd)\right| < h(d)$$

for a certain function $h(d)$. They showed $h(d) < d!$. Our main objective is to improve on this bound by showing

$$h(d) < d^{(1+\varepsilon)\log d}.$$

(Here ε approaches zero while d tends to infinity.)

In fact, we shall prove a more general theorem answering a question of Erdős [2], [4] affirmatively. Erdős raised the following problem: Let $A_k = \{a_1^{(k)} < a_2^{(k)} < \ldots\}$, $k = 1, 2, \ldots$ be a sequence of infinite sets of positive integers. Does there exist a function $f(k)$ so that for a suitable $g(n) = \pm 1$

$$\max_m \left|\sum_{i=1}^m g(a_i^{(k)})\right| < f(k)?$$

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Theorem 1.

$$f(k) < k^{(1+\varepsilon)\log k/2}$$

(1) is an immediate consequence of Theorem 1, since $h(d) \leqq 2f\left(\sum_{i=1}^{d} i\right) = 2f(d(d+1)/2)$.

From a result of Roth [7] on the discrepancy of sequences relative to arithmetic progressions it follow that $d^{1/2} < h(d)$, and the standard "random sequence" argument gives $d^{1/2} < f(d)$. There is a huge gap between the lower and upper bounds. The lower estimates appear to be more accurate. In fact, we suspect that $f(d) < d^c$ for some constant c.

In Section 2 we prove Theorem 1. In Section 3 we mention some generalizations and outline their proofs.

2. The prcof

Let $\varrho(d, M)$ be the smallest value of t such that the following holds: Given any t integral vectors $\mathbf{a}_1, \ldots, \mathbf{a}_t$ of dimension d, each having norm $|\mathbf{a}_i|_\infty \leqq M$ one can find a non-empty subset $H \subset \{1, \ldots, t\}$ and signs $\{\delta_i\}_{i \in H}$, $\delta_i = \pm 1$, so that $\sum_{i \in H} \delta_i \mathbf{a}_i = \mathbf{0}$.

The following simple lemma forms the core of the proof (see also Olson and Spencer [6] Section 2).

Lemma 1. $\varrho(d, M) \leqq 2d \log(dM)$.

Proof. We consider all sums of the form $\sum_{i \in I} \mathbf{a}_i$, where I is a subset of the interval $\{1, \ldots, t\}$. There are 2^t such sums. Each sum is a vector of the form $(b_1, \ldots, b_d)$, where $|b_i| \leqq tM$, hence there are at most $(2tM+1)^d$ distinct vectors among them. Our condition $t \geqq 2d \log(dM)$ implies $2^t > (2tM+1)^d$. We conclude by pigeonhole principle that there are two different subsets I and J of $\{1, \ldots, t\}$ such that $\sum_{i \in I} \mathbf{a}_i = \sum_{i \in J} \mathbf{a}_i$. We can complete the proof of the lemma by choosing $H = (i \setminus j) \cup (J \setminus I)$, $\delta_i = 1$ if $i \in I \setminus J$ and $\delta_i = -1$ if $i \in J \setminus I$. ∎

Next, we express Theorem 1 in terms of the incidence matrix of the sequences A_k. We define the (infinite) incidence matrix

$$V = V(A_1, A_2, \ldots) = [v_{k,s}]_{k=1}^{\infty}{}_{s=1}^{\infty}$$

by setting $v_{k,s} = 1$ if $s \in A_k$, else $v_{k,s} = 0$. Now Theorem 1 is equivalent to the statement that for an arbitrary infinite $0-1$ matrix $v = [v_{k,s}]_{k=1}^{\infty}{}_{s=1}^{\infty}$ there are signs $\varepsilon_1, \varepsilon_2, \ldots; \varepsilon_s = \pm 1$such that for every m

$$\left|\sum_{s=1}^{m} \varepsilon_s v_{k,s}\right| \leqq k^{(1+\varepsilon)\log k/2} \quad \textit{if} \quad k \geqq k_0(\varepsilon). \tag{2}$$

Observe that it suffices to prove (2) for $0-1$ matrices having arbitrary large *finite* number of rows. Indeed, assume that (2) have already been proved for $0-1$ matrices having n_r rows, with $n_r \to \infty$ if $r \to \infty$. That is, for every positive integer r there exists an array of signs $\{\varepsilon_s(r)\}_{s=1}^{\infty}$ satisfying (2) for $k_0(\varepsilon) \leqq k \leqq n_r$. Since we have only two signs, in the sequence $\{\varepsilon_1(r)\}_{r=1}^{\infty}$ there will one occuring infinitely many times, say $\varepsilon_1(r_{1,1})=\varepsilon_1(r_{1,2})=\dots$. Denote this common sign by ε_1.

Again, we will find an infinite subsequence $r_{2,1}, r_{2,2}, \dots$ of $r_{1,1}, r_{1,2}, \dots$ such that $\varepsilon_2(r_{2,1})=\varepsilon_2(r_{2,2})=\dots$. Denote this common sign by ε_2, and so on. Finally, we obtain an infinite array of signs $\{\varepsilon_s\}_{s=1}^{\infty}$ satisfying (2) for every $k \geqq k_0(\varepsilon)$.

It will be convenient for us to prove (2) for $0-1$ matrices having $n_r=2^r-1$ rows.

We start with some definitions.

Let $\mathscr{H}=\{H(i,j)\}_{i=1}^{r}{}_{j=1}^{\infty}$ be a family of finite subsets of $\mathbf{N}$ with the following properties.

(3a) $H(i,j_1)$ and $H(i,j_2)$ are disjoint if $j_1 \neq j_2$.

(3b) There are uniform bounds $t_1, \dots, t_r$ such that $|H(i,j)| \leqq t_i$.

(3c) For every m and i the interval $\{1, \dots, m\}$ can be written in the form $\bigcup_{1 \leqq j \leqq m_i} H(i,j) \cup P(i,m)$ for some m_i and $P(i,m)$ with $|P(i,m)| \leqq t_i - 1$.

A set-system $\mathscr{H}=\{H(i,j)\}_{1 \leqq i \leqq r,\, 1 \leqq j < \infty}$ having these properties will be called a $(t_1, \dots, t_r)$-system.

Associate a sign $\delta_n(i)=\pm 1$ with each $n \in \bigcup_{j=1}^{\infty} H(i,j)$, $1 \leqq i \leqq r$ and let $\mathscr{D}$ denote the array of signs, i.e.

$$\mathscr{D} = \{\delta_n(i)\}_{n \in H(i),\, 1 \leqq i \leqq r}, \quad \text{where} \quad H(i) = \bigcup_{j=1}^{\infty} H(i,j).$$

Let us be given a $(t_1, \dots, t_r)$-system $\mathscr{H}$ and an array of associated signs $\mathscr{D}$. We say that the matrices

$$V(i) = [v_{k,s}(i)]_{1 \leqq k \leqq 2^r-1,\, 1 \leqq s < \infty}, \quad 1 \leqq i \leqq r$$

are induced by $\mathscr{H}$ and $\mathscr{D}$, if the following recursion hold:

(4a) $V(0)=V$ (that is, $V(0)$ is the incidence matrix of the sequences A_k, $k=1, 2, \dots$).

(4b) $v_{k,s}(i) = \sum_{n \in H(i,s)} \delta_n(i) v_{k,n}(i-1)$ for $1 \leqq i \leqq r$, $1 \leqq k \leqq 2^r-1$, $1 \leqq s < \infty$.

Lemma 2. *There exist a $(t_1, \dots, t_r)$-system $\mathscr{H}$ and an array of associated signs $\mathscr{D}$ such that*

(5a) *the parameters $t_1, \dots, t_r$ satisfy the recursion $t_0=1$, $t_i=\varrho(2^{i-1}, t_0 t_1 \cdot \dots \cdot t_{i-1})$, $1 \leqq i \leqq r$;*

(5b) *in the induced matrices $V(i)=V(i, \mathscr{H}, \mathscr{D})$, $1 \leqq i \leqq r$, the first 2^i-1 rows are identically $\mathbf{0}$, i.e. $v_{k,s}(i)=0$ for $1 \leqq k \leqq 2^i-1$, $1 \leqq s < \infty$.*

We postpone she proof of Lemma 2 to the end of this section.

Now we deduce Theorem 1 from the lemmas. We prove that for an arbitrary $0-1$-matrix. $V=[v_{k,s}]_{1 \leqq k \leqq 2^r-1,\, 1 \leqq s < \infty}$ there exists a sequence of signs $\varepsilon_s=\pm 1$ such that for every m and $k_0(\varepsilon) \leqq k \leqq 2^r-1$ (2) holds.

By the application of Lemma 2 we obtain the existence of a $(t_1, \dots, t_r)$-system $\mathscr{H}=\{H(i,j)\}_{1 \leqq i \leqq r,\, 1 \leqq j < \infty}$ and an array of signs $\mathscr{D}=\{\delta_n(i)\}_{n \in H(i),\, 1 \leqq i \leqq r}$ satisfying (5a) and (5b). Let us define the sets $K(i,j)$ by the following formula:

$$K(1,j) = H(1,j)$$

$$K(i,j) = \bigcup_{n\in H(i,j)} K(i-1,n) \quad \text{for} \quad 2 \leqq i \leqq r.$$

Similarly, let (see (3c))

$$R(1,m) = P(1,m)$$

$$R(i,m) = \bigcup_{n\in P(i,m)} K(i-1,n) \quad \text{for} \quad 2 \leqq i \leqq r.$$

From the construction follows (see (3b) and (3c))

(6) $$|K(i,j)| \leqq t_1\cdot\ldots\cdot t_i \quad \text{and} \quad |R(i,m)| \leqq t_1\cdot\ldots\cdot t_{i-1}(t_i-1).$$

Finally, set

$$S(i) = \coprod_{j=1}^{\infty} K(i,j) \quad \text{for} \quad 1 \leqq i \leqq r, \; S(0) = \mathbf{N} \quad \text{and} \quad S(r+1) = \emptyset.$$

Obviously $S(0)\supset S(1)\supset\ldots\supset S(r+1)$.

Now we are ready to define the desired signs ε_s. If $s\in S(i)\setminus S(i+1)$ $(1\leqq i\leqq r)$, then there are indices s_j, $j=1, \ldots, i$ so that

$$s\in \bigcap_{j=1}^{i} K(j, s_j)$$

and let $\varepsilon_s = \bigcup_{j=1}^{i} \delta_{s_j}(j)$. If $s\in\mathbf{N}\setminus S_1$, then ε_s may be chosen arbitrarily.

From the definitions above directly follows the fundamental

(7) $$\sum_{n\in K(i,j)} \varepsilon_n v_{k,n} = \pm v_{k,j}(i).$$

We are now in the position to prove the upper bound in (2). Assume that $2^{q-1}\leqq k<2^q$. By repeated application of (3c) we obtain

$$\{1, \ldots, m\} = \bigcup_{1\leqq j\leqq m_1} H(1,j)\cup P(1,m_0) \quad (\text{let } m_0 = m),$$

$$\{1, \ldots, m_1\} = \bigcup_{1\leqq j\leqq m_2} H(2,j)\cup P(2,m_1), \ldots$$

$$\{1, \ldots, m_{q-1}\} = \bigcup_{1\leqq j\leqq m_q} H(q,j)\cup P(q,m_{q-1});$$

from which there follows

$$\{1, \ldots, m\} = \bigcup_{1\leqq j\leqq m_q} K(q,j)\cup R(q,m_{q-1})\cup\ldots\cup R(1,m_0).$$

Hence

$$\sum_{s=1}^{m} \varepsilon_s v_{k,s} = \sum_{j=1}^{m_q}\sum_{s\in K(q,j)} \varepsilon_s v_{k,s} + \sum_{i=1}^{q}\sum_{s\in R(i,m_{i-1})} \varepsilon_s v_{k,s}.$$

By (5b) and (7)

$$\sum_{s\in K(q,j)} \varepsilon_s v_{k,s} = \pm v_{k,j}(q) = 0$$

since $k<2^q$. Therefore, using (6)

$$\left|\sum_{s=1}^{m} \varepsilon_s v_{k,s}\right| = \left|\sum_{i=1}^{q} \sum_{s\in R(i,m_{i-1})} \varepsilon_s v_{k,s}\right| \leqq \sum_{i=1}^{q} |R(i, m_{i-1})| \leqq \tag{8}$$

$$\leqq \sum_{i=1}^{q} t_i \cdot \ldots \cdot t_{i-1}(t_i-1)| < t_1 \cdot \ldots \cdot t_q.$$

By Lemma 1 we have $\varrho(d, M) \leqq 2d \log (dM)$. Now an easy computation (using (5a)) yields

$$t_i \leqq 2^{i+c\sqrt{i}}$$

with some universal constant c. Returning to (8) we obtain

$$\left|\sum_{s=1}^{m} \varepsilon_s v_{k,s}\right| < t_1 \cdot \ldots \cdot t_q \leqq \prod_{i=1}^{q} 2^{i+c\sqrt{i}} \leqq 2^{\frac{q(q+1)}{2}+cq^{3/2}} \leqq k^{(1+\varepsilon)\log k/2}$$

for $k \geqq k_0(\varepsilon)$ since $2^{q-1} \leqq k$. This completes the deduction of Theorem 1 from lemmas.

Proof of Lemma 2. We shall construct the desired $\mathscr{H}=\{H(i,j)\}_{1\leqq i\leqq r,\, 1\leqq j<\infty}$ and $\mathscr{D}=\{\delta_n(i)\}_{n\in H(i),\, 1\leqq i\leqq r}$ by induction on i. Assume that $\mathscr{H}_i=\{H(h,j)\}_{0\leqq h\leqq i,\, 1\leqq j<\infty}$ and $\mathscr{D}_i=\{\delta_n(h)\}_{n\in H(i),\, 0\leqq h\leqq i}$ have already been defined so that $\mathscr{H}_i$ is a $(t_0, t_1, \ldots, t_i)$-system, the parameters $t_0, t_1, \ldots, t_i$ satisfy the recursion (5a) and the first 2^i-1 rows of the induced matrix $V(i)$ are identically $\mathbf{0}$. Let $H(0,j)=\{j\}$, $\delta_n(0)\equiv +1$ and $t_0=1$. Consider the 2^i-dimensional vectors $\mathbf{a}_s=(a_s^{(1)}, a_s^{(2)}, \ldots, a_s^{(2^i)})$ with coordinates $a_s^{(j)}=v_{l,s}(i)$, where $l=2^i+j-1$, $1\leqq j\leqq 2^i$.

By (6) and (7)

$$|\mathbf{a}_s|_\infty \leqq |K(i,s)| \leqq t_1 \cdot \ldots \cdot t_i,$$

thus, by the definition of $\varrho(d, M)$ one can select a non-empty subset $H(i+1, 1) \subset \subset \{1, \ldots, \varrho(2^i, t_1 \cdot \ldots \cdot t_i)\}$ and signs $\{\delta_n(i+1)\}_{n\in H(i+1,1)}$ such that

$$\sum_{n\in H(i+1,1)} \delta_n(i+1)\mathbf{a}_n = \mathbf{0}.$$

For an infinite subset B of $\mathbf{N}$ let $B[\varrho]$ denote the set of the ϱ smallest elements of B, i.e.

$$B[\varrho] = \{b_1, \ldots, b_\varrho\}, \quad \text{where} \quad B = \{b_1 < b_2 < \ldots\}.$$

Assume that $H(i+1,j)$, $1\leqq j\leqq p$ and the associated signs $\{\delta_n(i+1)\}_{n\in H(i+1,j)}$, $1\leqq j\leqq p$ have already been defined. Then, similarly as above, one can find a subset $H(i+1,p+1)$ of $B[\varrho]$, where $B=\mathbf{N}\setminus \bigcup_{j=1}^{p} H(i+1,j)$ and $\varrho=\varrho(2^i, t_1 \cdot \ldots \cdot t_i)$, and associated signs $\{\delta_n(i+1)\}_{n\in H(i+1,p+1)}$, $\delta_n(i+1)=\pm 1$ such that

$$\sum_{n\in H(i+1,p+1)} \delta_n(i+1)\mathbf{a}_n = \mathbf{0}. \tag{9}$$

(9) means that $v_{k,s}(i+1)=0$ for $2^i\leqq k\leqq 2^{i+1}-1$. $s=1,2,\ldots,$ that is, the first $2^{i+1}-1$ rows of $V(i+1)$ are identically $\mathbf{0}$ (the first 2^i-1 rows are $\mathbf{0}$ automatically). It is easy to see that $\mathscr{H}_{i+1}=\{H(h,j)\}_{0\leqq h\leqq i+1,\, 1\leqq j<\infty}$ is a $(t_0, t_1, \ldots, t_{i+1})$-system with $t_{i+1}=\varrho(2^i, t_1 \cdot \ldots \cdot t_i)$, thus, the induction step is complete. This proves Lemma 2, and thereby Theorem 1. ∎

3. Generalizations

We may reformulate Theorem 1 as follows. Given integer sequences $A_1, A_2, \dots$ it is possible to partition $\mathbf{N}$ into two parts N_1 and N_2 in such a way that, for each k and n, $A_k \cap N_1 \cap \{1, \dots, n\}$ and $A_k \cap N_2 \cap \{1, \dots, n\}$ contain approximately the same number of elements.

Now let us consider the following related question: What is the "smallest" function $f_p(k)$ $(p \geqq 3)$ such that, given integer sequences $A_1, A_2, \dots$, one can find a p-partition $N_1, \dots, N_p$ of $\mathbf{N}$ so that

$$\|A_k \cap N_i \cap \{1, \dots, n\}| - |A_k \cap N_j \cap \{1, \dots, n\}\| < f_p(k)$$

for all $1 \leqq i < j \leqq p$ and $k = 1, 2, \dots$?

Theorem 2.

$$f_p(k) < k^{(1+\varepsilon)\log k} \quad \textit{for} \quad k \geqq k_0(\varepsilon, p).$$

The proof of Theorem 2 goes along the lines as of the proof of Theorem 1, but instead of Lemma 1 we need Lemma 3 below. Details are left to the reader.

Lemma 3. *Let* $\mathbf{a}_1, \dots, \mathbf{a}_t$ *be d-dimensional integral vectors and let* $|\mathbf{a}_i|_\infty \leqq M$. *If* $d \geqq d_0(p)$ *and* $t \geqq 4d^2 \log^2(dM)$, *then one can select p pairwise disjoint non-empty subsets* $H_1, \dots, H_p$ *of* $\{1, \dots, t\}$ *such that* $\sum_{i \in H_j} \mathbf{a}_i = \sum_{i \in H_k} \mathbf{a}_i$ *for all* $1 \leqq j \leqq k \leqq p$.

Proof. Consider all sums of the form $\sum_{i \in I} \mathbf{a}_i$, where I is a subset of $\{1, \dots, t\}$ having cardinality $q = [\sqrt{t}]$ (integral part). There are $\binom{t}{q}$ such sums. The maximum norm of each sum is bounded by qM, hence there are at most $(2qM+1)^d$ distinct vectors among them. Our conditions $d \geqq d_0(p)$ and $t \geqq 4d^2 \log^2(dM)$ imply

$$\binom{t}{q} > q!(p-1)^q (2qM+1)^d.$$

By the pigeonhole principle there are $n > q!(p-1)^q$ subsets $I_1, \dots, I_n$ such that all sums $\sum_{i \in I_j} \mathbf{a}_i$, $1 \leqq j \leqq n$ are equal. Applying a well-known theorem of Erdős and Rado [5] to the set-system $\{I_j\}_{j=1}^n$ we obtain that one can select p of them $J_1, \dots, J_p$ which form a strong Δ-system, that is, the intersection of any two J_i's is the same set. Denote it by $D = J_1 \cap J_2$ and set $H_i = J_i \setminus D$, $1 \leqq i \leqq p$. ∎

We can express Theorem 1 in terms of vectors as follows: Let $\mathbf{a}_1, \mathbf{a}_2, \dots$ be infinite dimensional $0-1$ vectors, then one can find signs $\varepsilon_1, \varepsilon_2, \dots$; $\varepsilon_i = \pm 1$, and a vector $\mathbf{w}$ such that for every m

$$\sum_{i=1}^{m} \varepsilon_i \mathbf{a}_i \leqq \mathbf{w}, \tag{10}$$

where $\mathbf{w} = (w^{(1)}, w^{(2)}, \dots)$ and $w^{(k)} = k^{(1+\varepsilon)\log k/2}$ for $k \geqq k_0(\varepsilon)$. Here $\mathbf{a} \leqq \mathbf{w}$ means that $|a^{(i)}| \leqq w^{(i)}$, where $\mathbf{a} = (a^{(1)}, a^{(2)}, \dots)$.

Finally, we mention the "continuous" version of (10).

Theorem 3. *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots$ *be infinite dimensional vectors satisfying* $|\mathbf{v}_i|_\infty \leqq 1$. *Then one can find signs* $\varepsilon_1, \varepsilon_2, \ldots; \varepsilon_i = \pm 1$, *and a vector* $\mathbf{u}$ *such that for every* m

$$\sum_{i=1}^{m} \varepsilon_i \mathbf{v}_i \leqq \mathbf{u},$$

where $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots)$, $u^{(k)} = k^{(2+\varepsilon)\log k}$ *for* $k \geqq k_1(\varepsilon)$.

Theorem 3 is an infinite dimensional version of the following result of Bárány and Grinberg [1]: Given $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{R}^d$ satisfying $|\mathbf{v}_i|_\infty \leqq 1$, there exists $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ such that

$$\max_{1 \leqq m \leqq n} \left| \sum_{i=1}^{m} \varepsilon_i \mathbf{v}_i \right|_\infty \leqq 2d.$$

Proof. We deduce Theorem 3 from (10). In fact, we shall use the following slight generalization of (10): Given infinite dimensional vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots$ having coordinates 0, 1 or -1, there exist $\varepsilon_1, \varepsilon_2, \ldots; \varepsilon_i = \pm 1$ such that

$$\max_{m} \left| \sum_{i=1}^{m} \varepsilon_i \mathbf{a}_i \right| \leqq \mathbf{w} \tag{11}$$

with $\mathbf{w} = (w^{(1)}, w^{(2)}, \ldots)$, $w^{(k)} = k^{(1+\varepsilon)\log k/2}$ for $k \geqq k_2(\varepsilon)$. Its proof is left to the reader.

Denote the j-th coordinate of $\mathbf{v}_i$ by $v_i^{(j)}$. Since $-1 \leqq v_i^{(j)} \leqq 1$, thus it is reprezentable in the form

$$v_i^{(j)} = \sum_{s=0}^{\infty} v(i, j, s) 3^{-s}, \tag{12}$$

where $v(i, j, s) = 1$ or -1 or 0. Define a bijection β: $\mathbf{N} \times (\mathbf{N} \cup \{0\}) \to \mathbf{N}$ as follows:

$$\beta(j, s) = \binom{j+s}{2} + j.$$

Set $a_i^{(k)} = v(i, j, s)$, where $(j, s) = \beta^{-1}(k)$. Furthermore, let $\mathbf{a}_i = (a_i^{(1)}, a_i^{(2)}, \ldots)$, $i = 1, 2, \ldots$. By the application of (11) we obtain that there exist $\varepsilon_1, \varepsilon_2, \ldots; \varepsilon_i = \pm 1$ such that

$$\max_{m} \left| \sum_{i=0}^{m} \varepsilon_i a_i^{(k)} \right| \leqq k^{(1+\delta)\log k/2} \quad \text{for} \quad k \geqq k_2(\delta),$$

or equivalently,

$$\max_{m} \left| \sum_{i=1}^{m} \varepsilon_i v(i, j, s) \right| \leqq k^{(1+\delta)\log k/2} \tag{13}$$

with $k = \beta(j, s)$ and $k \geqq k_2(\delta)$. Multiplying (13) by 3^{-s} and summing for $s = 0, 1, \ldots$ we have (see (12))

$$\max_{m} \left| \sum_{i=1}^{m} \varepsilon_i v_i^{(j)} \right| \leqq \sum_{s=0}^{\infty} \exp\{(1+\delta) \log^2 \beta(j, s)/2\} 3^{-s} \tag{14}$$

for $j \geqq j_2(\delta)$. A simple calculation yields that

$$\sum_{s=0}^{\infty} \exp\{(1+\delta) \log^2 \beta(j, s)/2\} 3^{-s} \leqq j^{(2+\varepsilon)\log j},$$

where $\varepsilon=\varepsilon(\delta)\to 0$ if $\delta\to 0$. Returning to (14) we obtain that, for sufficiently large j

$$\max_m \left| \sum_{i=1}^{m} \varepsilon_i v_i^{(j)} \right| \leqq j^{(2+\varepsilon)\log j},$$

which was to be proved. ∎

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